Chapter 1
Financial Bubbles

Efsun Kürüm, Gerhard-Wilhelm Weber and Cem İyigün

Abstract We study on speculative financial bubbles whose characteristic can be modeled by Log Periodic Power Law (LPPL) which is represented by D. Sornette and his colleagues. The most probable time of the crash is estimated by a parameter in the equation. All parameters used in the equation, are forecasted by optimization through a genetic algorithm. Analysis of a time series by S&P 500 from 1987 shows the signals of the LPPL before the financial crisis of October 1987. In addition to the speculative bubbles, we also present and investigate antibubbles. They, likewise speculative bubbles, also follow log-periodic power law but, of course, with decelerating oscillations and, generally, in a bearish way inclined instead of a bullish way. We also introduce an alternative method which approaches the bubble concept geometrically and benefits from the advantages of optimization and machine learning.

Keywords: Financial mathematics, bubble, Log-periodic power law, optimization, fitting, prediction.

1.1 Introduction

Basically, financial bubbles are artificial increases in price which are created by high expectations of individuals; in other words, artificial price augmentation originated from greediness of human beings. Since some investors do not know how to manage their risk when the bubbles burst, this causes big disappointment resulting even in

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committing suicide. Moreover, their results affect the whole world like prompting domino stones.

For instance, because of the recent real-estate bubble and the subsequent subprime crises, 4 trillion dollar global cumulative losses of financial institutions have occurred and more than 6 million job losses happened in America up to 2009 [25]. It is still very difficult to characterize, to estimate and to prevent them from further inflating in advance.

1.1.1 What are Bubbles?

A bubble is said to occur if an asset price goes beyond its *fundamental value*. Here, the adversity is to determine the fundamental value of an asset. This value of an asset is arranged in equilibrium, endogenously. It is generally not given, i.e. , not exogeneously. The fundamental value determines whether a bubble took place at all and which component of the price is due to a bubble [3].

An investor is willing to buy or hold an asset at a price which is above its fundamental value if he thinks that the asset can be resold at an even higher price in a later trading round [3]. This trading behaviour is called "speculation" by *Harrison* and *Kreps*, 1978 [10]. As a result, the fundamental value can be thought as the price which an investor is eager to pay if he is compelled to hold the asset forever, i.e., if he is not allowed to retrade [3].

However, the fundamental value is generally insufficient constrained and it is not possible to distinguish between an exponentially growing fundamental price and an exponentially growing bubble price. Therefore, a more precise definition of bubbles is required to detect them [25]. In the Conclusion and Outlook part, Section 5, alternative approaches will be introduced for that purpose.

1.1.2 Why Bubbles Matter?

When any bubble goes burst, some - in fact, often many - people lose. Sometimes, experienced speculators can realize the end of a bubble and cut their positions in time, since they generally know how to limit their risk to a bearable level. The investors with very little experience of how to manage risks hold the asset at the late phase of a bubble; therefore, they are damaged by the bubbles excessively [4].

If a bubble affected only a few investors, it would be of a limited importance because it has little effect on the overall economy. Bubbles can cause major problems when they originate in an asset that is commonly held. Then, when the bubble bursts, not only a large number of people suffer directly, but it also interacts with the economy. Therefore, dangerous bubbles are generally the ones in a stock market [4].

1.2 Famous Historical Bubbles

1.2.1 Tulip Mania

The tulip mania is considered as the first recorded financial bubble; occured in the 1630s in Holland. In 1593, tulips were introduced by a botanist *Carolus Clusius* who brought them from Ottomon Empire. He planted his garden to examine them for medicinal purposes. Then, his neighbours stole some of his bulbs in order to make some quick money. Hence, they gave rise to start the process of Dutch bulb trade [24].

In 1634, the madness of having tulips became too much and daily work was neglected. Even the lowest members of the society joined the tulip trade. Until 1635, the mania ascended, the prices increased and many people invested fortunes to own tulip bulbs. Then, to sell tulips by their weight, in *perits* became necessity. Perit was a small weight which is less than a grain. In fact, 480 grains equalled 1 ounce. Prices for different varieties were as follows:

Admiral Liefken, weighing 400 perits = 4,400 florins; *Admiral Von der Eyk*, weighing 446 perits = 1,260 florins; *Shilder* of 106 perits = 1,615 florins; *Viceroy* of 400 perits = 3,000 florins; *Semper Augustus*, weighing 200 perits = 5,500 florins [22].

To perceive, the value of a single tulip, Table 1.1 can be useful. It was recorded in terms of items and their values by one of the authors of that time, *Munting*.

Table 1.1 Items and their values which were traded for a single Viceroy bulb [22]

Item	Value (florins)
Two lasts of wheat	448
Four lasts of rye	558
Four fat oxen	480
Eight fat swine	240
Twelve fat sheep	120
Two hogsheads of wine	70
Four casks of beer	32
Two tons of butter	192
A complete bed	100
A suit of clothes	80
A silver drinking cup	60
Total	**2,500**

As with all gambling mania, at the beginning, confidence was high because everybody was gaining. Tulip stocks were speculated in the rise and fall by the tulip-brokers. They earned much money, buying the stocks when prices fell, and selling when they rose. Many people suddenly became rich. Everyone imagined that this process would last forever. Eventually, wise investors began to recognize that this imaginary world could not last forever [22]. After that, as it happens in many cases

of speculative bubbles, some prudent people started to sell and freezed their profits. As everyone tried to sell while not many were buying, a domino effect became realized and prices became lower and lower. This caused people to panic and sell regardless of losses [2]. Confidence was destroyed and the tulip market collapsed in February 1637, abruptly as shown in Figure 1.1.

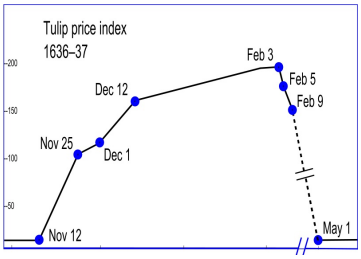


Fig. 1.1 A standardized price index for tulip bulb contracts [23].

1.2.2 South Sea Bubble

The South Sea Company was established in 1711 by Earl of Oxford *Harley*. As a result of the war between Spain and Britain, British Government left 10 million pounds in debt. Harley offered the government paying back 10 million pounds of debt to ameliorate the government's financial condition. In turn, the government proposed %6 interest to the company [21]. In addition to 6% interest, the government suggested privileged trading rights with Spain' s colonies in the South Seas, today known as South America [5].

To finance its operations, the company issued stock to investors and the company's shares were snatched by investors, rapidly. After their first share issue success, the company issued more shares. Although the company had an inexperienced administration team, investors gathered this stock, competitively [5]. It was thought that this company "could never fail".

The war between Britain and Spain began again in 1718; therefore all trading occasions stopped. However, this event did not deter the investors for buying stocks. After a while, management of the South Sea Company recognized that they were not generating much profit from company's operations. For this reason, to generate revenues, they decided to place more importance on issuing stock instead of making actual trade. Meanwhile, the leaders of the company decided to sell their shares since they noticed that the company's stocks were fabulously overvalued relative to its profit. At that time, the other investors did not realize that, actually, the firm was scarcely profitable [5].

Eventually, rumors spread among investors about the firm's management team had sold all of their shares in the company. Then, panic selling immediately started

and soon the South Sea Company's shares became worthless as is evident from the Figure 1.2. *Isaac Newton* lost over 20,000 pounds. He stated "I can calculate the motion of heavenly bodies, but not the madness of people" [5]. After the South Sea Bubble busted, Britain's economy collapsed in spite of the government's endeavors and to heal completely almost took a century.



Fig. 1.2 Log scale price for south sea company stocks [7].

1.2.3 1929 Great Depression

The end of World War I brought a new era to America. An era of confidence, optimism and welfare were being experienced in the United States during 1920s. After World War I, industrialization and evolution of new technologies such as radio, automobile, air flight bolstered the economic and cultural boom. The Dow Jones Industrial Average, DJIA, increased throughout the 1920s and because of the country's strong economic conditions, most of the economists thought shares were the most confident investment. These caused many investors to buy stocks, greedily [6].

In a little while, investors purchased stocks *on margin* which means that the buyer would invest some of his own money, but the rest compensated by the broker. In those years, only 10 to 20% of the stock price had to be paid by the buyer and, hence, 80 to 90% of the cost of the stock price would be paid by the broker. If the price of the shares declined lower than the amount of the loan, the broker would probably issue a *margin call*, i.e., the buyer must pay back his loan as cash immediately. Therefore, to buy shares on margin could be very risky. However, in the 1920s, many people who expected to make a lot of money on the stock market easily, called speculators, purchase stocks on margin. They supposed that this rising process in

prices never end; so they could not recognize how serious the risk was which they were taking [17].

The Dow Jones bounced from 60 to 400 from 1921 to 1929. This generated a lot of new millionaires. Many people mortgaged their homes and invested their savings into stock market. However, few people really had knowledge about the companies in which they invested [6].

In 1929, from June through August, stocks prices reached their highest levels. Economist *Irving Fisher* stated that "Stock prices have reached what looks like a permanently high plateau", this was the statement many speculators wanted to hear and believe. On 3rd of September 1929, the DJIA closed at 381.17; hence, the stock market had reached its peak. Two days later, the market commenced falling. Stock prices fluctuated throughout September and into October [17].

As indicated in Figure 1.3, a strong bear market, a market condition in which the prices of shares were decreasing, had started by October 1929. On 24th of October 1929 which is known as *Black Thursday*, panic selling started since investors distinguished the stock market boom was actually an over inflated speculative bubble [6]. Rumors spread that people commit suicide [17]. Although the Federal Reserve Bank increased interest rates several times to relieve stock market and overheated economy in 1929, this could not prevent sad end.

When the stock market crashed on 28th and 29th of October, millionaire margin investors went bankrupt instantly. In November 1929, DJIA sharply declined from 400 to 145. Over $5 billion worth of market capitalization had been vanished from stocks that were trading on the New York Stock Exchange in just three days. The stock market crash of 1929 caused a great economic crisis known as the *Great Depression* [6].

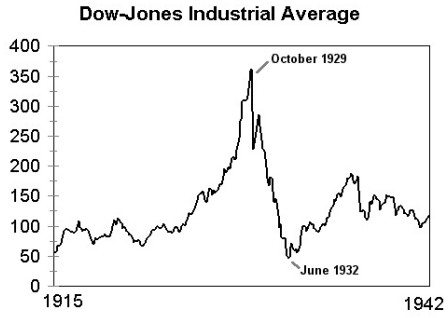


Fig. 1.3 Dow Jones Industrial Average index from 1915 to 1942 [20].

As we observe from the famous bubble examples above, irrational expectations always trigger destructive financial crashes.

Indeed, this big and yet vaguely understood phenomenon of "bubbles" asks for a high academic excellence. Academicians and central bankers still try to find a method to prevent from bubbles and try to develop a model which estimates bub-

bles, previously. One of the successful models which detects bubbles was developed by Professor *D. Sornette* and his colleagues, called *Johansen-Ledoit-Sornette* (*JLS*) *model*.

1.3 The Johansen-Ledoit-Sornette Model

At a symposium sponsored by the Federal Reserve Bank of Kansas City, in 2002, the former Federal Reserve chairman *Alan Greenspan* said: "... we recorganized that, despite our suspicions, it was very difficult to definitively identify a bubble until after the fact - that is, when its bursting confirmed its existence". In his speech, he declared that "It seems reasonable to generalize from our recent experience that no low-risk, low-cost, incremental monetary tightening exists that can reliably deflate a bubble. But is there some policy that can at least limit the size of a bubble and, hence, its destructive fallout? From the evidence to date, the answer appears to be *no*. But we do need to know more about the behavior of equity premiums and bubbles and their impact on economic activity" [9]. As we understood from his speech, there is no exact theory which can estimate the bubbles previously, and as he underlined that *how it* important to identify them before they blow up.

Didier Sornette and his colleagues have proposed a theoretical model, known as the *Johansen-Ledoit-Sornette (JLS)* model, to designate bubbles in advance [25]. By analyzing the cumulative human behaviors, the price dynamics is defined by the model during a time interval related with a bubble process (also called a *regime*). Also, after a bubble regime, the most probable crash time can be estimated by the JLS model previously. This model actually does not describe bubbles by exponential prices but rather by *faster-than-exponential growth* of price. The reason for this arises from imitation and herding behavior of noise traders, called as *irrational investors* [25].

1.3.1 Evolvement of the Price

In the JLS model, only a speculative asset is considered that does not pay dividends. The interest rate, risk aversion, information asymmetry, and the market-clearing condition are ignored for simplicity. Rational expectations are equivalent to the martingale hypothesis [19]:

$$E_t\left[p(t^{'})\right] = p(t) \quad \forall t^{'} > t. \tag{1.1}$$

Here, the price of the asset at time t and the expectation conditional on information up to time t are showed by $p(t)$ and $E_t[\cdot]$, respectively. The solution of Eqn. (1.1) is a constant, if the asset price is not permitted to fluctuate under the effect of noise as follows:

$$p(t) = p(t_0).$$

Here, t_0 displays some initial time and the meaning of this equality is that there is no change in the price. Since the asset price pays no dividend, without loss of generality, for simplicity, its fundamental value is $p(t) = 0$. Thus, a positive value of $p(t)$ generates a speculative bubble [13]. In the model, the crash is the jump, each jump is defined as a bubble and small jumps are not considered. A jump process can be indicated by j and the value zero is attained before the crash, and the value one after the crash, as representing in the following:

$$j = \begin{cases} 0, & \text{before crash,} \\ 1, & \text{after crash.} \end{cases}$$

As depicted in Figure 1.4, the cumulative distribution function (cdf) of the time of the crash, t_c, is called $1 - Q(t)$ and the probability density function (pdf) is

$$q(t) := \frac{dQ}{dt}(t) \approx \frac{\Delta Q}{\Delta t},$$

The hazard rate of the crash is defined as [13]:

$$h(t) = q(t)/[1 - Q(t)],$$

where $Q(t) := Q(t_c > t)$. The hazard rate $h(t)$ compares that change of probability with the likelihood $1 - Q(t) = Q(t_c \leq t)$.

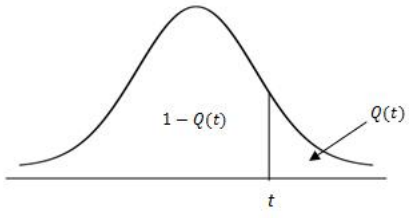


Fig. 1.4 The cumulative distribution function (cdf) of the time of the crash.

According to the JLS model, the asset price is accepted to drop by a certain percentage $\kappa \in (0,1)$ in case of a crash, and the dynamics of the asset price before the crash is given by [19]:

$$dp_t = \mu(t)p(t)dt - \kappa p(t). \tag{1.2}$$

In order to satisfy the martingale condition for the price process, in Eqn. (1.2), the time dependent drift, $\mu(t)$, is chosen, namely

$$E_t[dp_t] = \mu(t)p(t)dt - \kappa p(t)h(t)dt = 0$$

$$\Rightarrow \mu(t) = \kappa h(t). \tag{1.3}$$

Substituting Eqn. (1.3) into Eqn. (1.2) will give us Eqn. (1.4) below:

$$dp = \kappa h(t)p(t)dt - \kappa p(t)dj. \tag{1.4}$$

While $j = 0$ being the case before the crash, we get an ordinary differential equation, ODE:

$$dp = \kappa h(t)p(t)dt \Rightarrow \frac{dp}{dt} = \kappa h(t)p(t) \Rightarrow p^{'}(t) = \kappa h(t)p(t)$$

$$\Rightarrow \frac{p^{'}(t)}{p(t)} = \kappa h(t). \tag{1.5}$$

Integrating the Eqn. (1.5) from both sides gives

$$\int_{t_0}^{t} \frac{p^{'}(s)}{p(s)} ds = \int_{t_0}^{t} \kappa h(s) ds.$$

Thus, the following expression is obtained

$$\log\left[\frac{p(t)}{p(t_0)}\right] = \kappa \int_{t_0}^{t} h(s)ds \qquad \text{before the crash.} \tag{1.6}$$

According to this new statement, under the condition that there is no crash, any increase of the probability of a crash implies an increase of the price speed in order to fulfill the martingale property. With [13] we note that the hazard rate $h(t)$ of the crash drives the price; that rate is not restricted by any bound so far.

To clarify the investors' behaviours, the model considers a *network* of agents. Each of the *agents* is indexed by an integer $i = 1, 2, \ldots, I$, and the set of agents connected with i is indicated by $N(i)$. According to some graph like in Figure 1.5, the set of agents, $N(i)$, is directly linked with agent i. An assumption in the model is that agent i can have only one of two possible states, $s_i \in \{-1, +1\}$. Here, the values -1 and $+1$ can be interpreted as buy or sell, bullish or bearish or, in technical or Boolean terms, "on or off". The state of investor i is determined by the model as follows [19]:

$$s_i = \text{sign}\left(K \sum_{j \in N(i)} s_j + \sigma \varepsilon_i\right). \tag{1.7}$$

Here, K is a positive constant and ε_i is independently distributed with the standard normal distribution. The parameter K, called the *coupling strength*, controls the tendency towards imitation and σ governs the tendency towards *idiosyncratic* behavior. The parameter σ can be interpreted as an *environmental* factor. Bigger values of K

reflect *strong imitation* [13]. If $K = 0$, only environmental effects determine the decision of the trader.

The ratio K/σ represents the outcome of the order and disorder decision; in other words, it symbolizes the probability of a crash. Also, as long as the average of all values K was strictly positive, the model allows some of the values of K even to be negative. The meaning of the negative K is that the investor does not accept other investors' decisions.

As for the *susceptibility* of the system, it can be explained by adding a global influence term G to the Eqn. (1.7):

$$s_i = \text{sign}\left(K \sum_{j \in N(i)} s_j + \sigma\varepsilon_i + G\right). \tag{1.8}$$

In Eqn. (1.8), $K\sum_{j\in N(i)} s_j + \sigma\varepsilon_i$ corresponds to individual decision and G can be explained as an average group effect to the investor's decision.

The parameter G will converge to the state $+1$ if $G > 0$; G will approximate the favour state -1 if $G < 0$ and, if $G = 0$, no global influence will exist.

The average state is defined as $M = (1/I)\sum_{i=1}^{I} s_i$. When there is no global influence, expectation of the average state is zero, $E(M) = 0$, i.e., agents are equally divided between the two states.

Provided that a positive (negative) global influence is given, agents in the positive (negative) state will outnumber the others, $E(M) \cdot G \geq 0$. With this notation, the *susceptibility* of the system is defined as [13]:

$$\chi = \frac{d(E(M))}{dG}|_{G=0}. \tag{1.9}$$

The sensitivity of the average state to a small global influnce is measured by this susceptibility. It also has a second interpretation, namely, as the expected squared deviance (variance) of the average state M around its mean of zero, caused by the random idiosyncratic shocks ε_i.

The susceptibility is affected by the structure of the network. At the basis of the JLS model, there are two kinds of network structures: The 2-*dimensional grid* and the *hierarchical diamond lattice*.

1.3.2 Two-Dimensional Grid

In this network structure, the agents are placed on a 2-dimensional grid in the Euclidean plane. Each agent has 4 nearest neighbors: one to the North, one to the South, one to the East and one to the West. The related parameter K/σ evaluates the propensity towards imitation relative to the tendency towards idiosyncratic behavior [13].

The properties of the system are arranged by a critical point K_c. While $K < K_c$, *disorder* decision wins. The meaning of this, the sensitivity is small for a small

global influence, imitation only propagates, like in a *chain reaction* or even *cascading*, between close neighbors and the susceptibility χ of the system is finite.

When K rises and approaches K_c, *order* decision reigns, i.e., the system is extremely sensitive to a small global perturbation, imitation propagates over long distances. In this situation, the susceptibility χ of the system goes to infinity with respect to the power law. This behaviour of the χ can be explained mathematically as follows [19] :

$$\chi \approx A(K_c - K)^{-\gamma}, \qquad \gamma > 1. \tag{1.10}$$

Here, the parameter K does not even need to be deterministic, but it could also be a stochastic process, as long as it proceeds slowly enough. The value t_c is called the *first time* such that $K(t_c) = K_c$. Then, before the critical date t_c, the following approximation is obtained from Taylor expansion:

$$K_c - K(t) \approx \text{constant} \cdot (t_c - t).$$

Since the hazard rate of the crash behaves similar to the susceptibility around the critical point, this approximation yields the following statement:

$$h(t) \approx B \cdot (t_c - t)^{-\alpha}. \tag{1.11}$$

Here, B is a positive constant, and if the bubble has not bursted yet, to prevent the price from going to infinity when approaching t_c, α should be between 0 and 1. Since the crash could happen at some time prior to t_c, although the crash is not most probably, as stated in [13], t_c is not necessarily the time of the crash. Finally, substituting Eqn. (1.11) into Eqn. (1.6) gives us the following relation:

$$\log p(t) = \log p(t_c) + \kappa B \int_{t_c}^{t} (t_c - s)^{-\alpha} ds,$$

i.e.,

$$\log p(t) = \log p(t_c) - \frac{\kappa B}{1-\alpha}(t_c - t)^{1-\alpha}.$$

Hence,

$$\log p(t) = \log p(t_c) - \frac{\kappa B}{\beta}(t_c - t)^{\beta} \quad \text{before the crash.} \tag{1.12}$$

Since $\alpha \in (0,1)$, $\beta := 1 - \alpha \in (0,1)$ and $p(t_c)$ is the price at the critical time t_c. The logarithm of the price before the crash also obeys the power law. As the price process approaches the critical date, the slope of the log price (the expected return per unit of time) becomes limitless. This helps to compansate boundless probability of the crash in the next instant [19].

1.3.3 Hierarchical Diamond Lattice

Another network structure is the hierarchical diamond lattice. To obtain this lattice, the process starts with a pair of investors who are connected to each other. Then, these connections are changed according to a *diamond* structure. On the diamond, two original investors reside two opposite vertices and where the two new investors occupy on the other two vertices. There are 4 links included by this diamond. For each one of these 4 links, it is altered by a diamond in the aforementioned way, and this operation is iterated [18]. Then, a diamond lattice is formed, as illustrated in Figure 1.5.

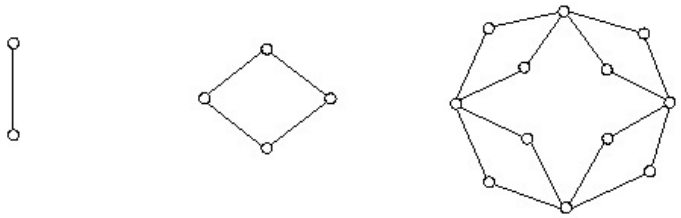


Fig. 1.5 The relationship between traders and links.

In Figure 1.5, circles and edges show traders and links, respectively. After p iterations, $(2/3)(2+4^p)$ investors and 4^p connections between them are obtained [18]. This can be proven by induction principle.

The basic properties of the hierarchical diamond lattice are similar to the 2-dimensional grid. In this network structure, there is also a critical point K_c. The susceptibility is restricted while $K < K_c$, and it goes to infinity as K increases towards K_c. Unlike from 2-dimensional grid, here, in the hierarchical diamond lattice, the critical exponent can be a complex number. The general solution for the susceptibility is a sum of terms like the one in $\chi \approx A(K_c-K)^{-\gamma}$, but now with complex exponents $-\gamma+iw$, etc. [13]. This approximation is evaluated gradually as follows:

$$\begin{aligned}
\chi &\approx \mathrm{Re}\left[A_0(K_c-K)^{-\gamma}+A_1(K_c-K)^{-\gamma+iw}+\ldots\right], \qquad (1.13)\\
\chi &\approx \mathrm{Re}\left[A_0(K_c-K)^{-\gamma}\right]+\mathrm{Re}\left[A_1(K_c-K)^{-\gamma+iw}\right]+\ldots,\\
\chi &\approx \mathrm{Re}\left[A_0(K_c-K)^{-\gamma}\right]+\mathrm{Re}\left[A_1(K_c-K)^{-\gamma}(K_c-K)^{iw}\right]+\ldots,\\
\chi &\approx \mathrm{Re}\left[A_0(K_c-K)^{-\gamma}\right]+\mathrm{Re}\left[A_1(K_c-K)^{-\gamma}e^{iw\log(K_c-K)}\right]+\ldots,\\
\chi &\approx A_0(K_c-K)^{-\gamma}+A_1(K_c-K)^{-\gamma}\mathrm{Re}\left[e^{iw\log(K_c-K)}\right]+\ldots,\\
\chi &\approx A_0(K_c-K)^{-\gamma}+\\
&\quad A_1(K_c-K)^{-\gamma}\mathrm{Re}\left[\cos(w\log(K_c-K))+i\sin(w\log(K_c-K))\right]+\ldots,
\end{aligned}$$

$$\chi \approx A_0(K_c-K)^{-\gamma}+A_1(K_c-K)^{-\gamma}\cos\left[w\log(K_c-K)+\psi\right]+\ldots.$$

Here, Re$[\cdot]$ indicates the real part of a complex number and γ, A_0, A_1, w, ψ are real numbers. Hence, the *power law* is corrected by oscillations whose frequency busts as the price process approach the critical time. These accelerating oscillations are called log-periodic, and their frequency are $\lambda = w/2\pi$ is called the log-frequency [13]. If the same steps are applied, Eqn. (1.13) is obtained for the hazard rate of a crash:

$$h(t) \approx B_0(t_c-t)^{-\alpha}+B_1(t_c-t)^{-\alpha}\cos\left[w\log(t_c-t)+\psi'\right]. \tag{1.14}$$

Substituting Eqn. (1.14) into Eqn. (1.12), we get:

$$\log\left[p(t)\right] = \log\left[p(t_c)\right]-\frac{\kappa}{\beta}\left\{B_0(t_c-t)^{\beta}+B_1(t_c-t)^{\beta}\cos\left[w\log(t_c-t)+\phi\right]\right\}, \tag{1.15}$$

and (1.15) can be rewritten as

$$y(t) \approx A+B(t_c-t)^{\beta}+C(t_c-t)^{\beta}\cos\left[w\log(t_c-t)+\phi\right], \tag{1.16}$$

where $A=\log\left[p(t_c)\right]$, $B=-\kappa B_0/\beta$, $C=-\kappa B_1/\beta$ and ϕ is another phase constant.

1.4 The Fitting Process

In the fitting process, so as to diminish the number of unknown parameters, three linearly embedded variables, A, B, C, have been optimally adjusted and calculated from the obtained values of the nonlinear parameters. For this reason, approximate Eqn. (1.16) is rewritten as [11]

$$y(t) \approx A+Bf(t)+Cg(t), \tag{1.17}$$

where

$$f(t)=\begin{cases}(t_c-t)^{\beta}, & \text{for a speculative bubble,}\\ (t-t_c)^{\beta}, & \text{for an antibubble.}\end{cases}$$

and

$$g(t)=\begin{cases}(t_c-t)^{\beta}\cos\left(w\log(t_c-t)+\phi\right), & \text{for a speculative bubble,}\\ (t-t_c)^{\beta}\cos\left(w\log(t_c-t)+\phi\right), & \text{for an antibubble,}\end{cases}$$

An *antibubble* is the opposite of the speculative bubble. Like a speculative bubble, it also follows a log-periodic power law (LPPL) but, of course, with decelerating oscillations and generally being bearish inclined instead of bullish. The term antibubble comes from the term *antiparticle* in physics, since an antiparticle is sim-

ilar to its sibling particle, except that it carries opposite charges and demolishes its sibling particle when it comes across with it [12].

The best values of the linear parameters are calculated for each choice of the nonlinear parameters by applying ordinary least-squares (OLS) method as follows by Gaussian normal equations [11]:

$$Xb = y \;\Rightarrow\; (X^T X)b = X^T y.$$

$$\begin{pmatrix} N & \sum_{i=1}^N f(t_i) & \sum_{i=1}^N g(t_i) \\ \sum_{i=1}^N f(t_i) & \sum_{i=1}^N f(t_i)^2 & \sum_{i=1}^N f(t_i)g(t_i) \\ \sum_{i=1}^N g(t_i) & \sum_{i=1}^N f(t_i)g(t_i) & \sum_{i=1}^N g(t_i)^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N \log p_i \\ \sum_{i=1}^N \log p_i f(t_i) \\ \sum_{i=1}^N \log p_i g(t_i) \end{pmatrix},$$

where

$$X := N \begin{pmatrix} 1 & f(t_1) & g(t_1) \\ \vdots & \vdots & \vdots \\ 1 & f(t_N) & g(t_N) \end{pmatrix}, b := \begin{pmatrix} A \\ B \\ C \end{pmatrix} \text{ and } y := \begin{pmatrix} \log p_1 \\ \vdots \\ \log p_N \end{pmatrix}.$$

Here, t_i $(i = 1,2,\ldots,N)$ are the times of the price sampling (t_i, p_i). The general solution of this equation is given by

$$\hat{b} = (X^T X)^{-1} X^T y.$$

Of course, we may assume that the number of data is greater than to the number of unknows, p, i.e., $N \geq p = 3$, now the solution A, B and C become inserted into problem (1.17) and that the design matrix X has full rank. In case of ill-conditionedness, we may apply a regularization technique [1].

Fitting a function to data is a nonlinear estimation problem of the residuals sum of squares, RSS, where the objective function is given by [11]

$$\min_{\theta} \; F(\theta) := \sum_{i=1}^{N} \left(y_\theta(t_i) - y_i\right)^2;$$

Here, $\theta = (t_c, \phi, w, \beta)^T$ is the vector of unknown parameters, and $y_\theta(\cdot) := y(\cdot)$, depending on θ.

The function of the residual sum of squares, F, is an analytically complicated or a strongly nonconvex function, and it can comprise multiple local minima with quite similar values. Here, the aim is to find the global minimum and to optimize the objective function with methods like the *downhill-simplex method* or the *quasi-Newton method*, this could be hazardous since these methods can be tricky by directing a local minimum rather than the global minimum [11]. Therefore, only more model-free *global optimization* solution methods were considered, such as *Simulated Annealing*, *Taboo Search* and *Genetic Algorithm*, etc.. Here, we used the Genetic Algorithm [14, 15, 16], and then as an example of a speculative bubble, the fitting process for S&P 500 index from 1980 to 1988 was obtained as shown in Figure 1.6.

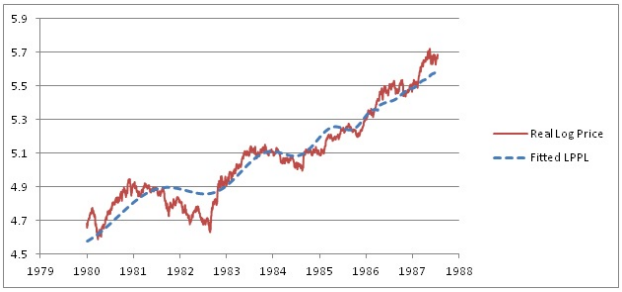


Fig. 1.6 The LPPL equation fitted to S&P 500 index.
Parameters: $A = 5.5829$, $B = -0.1973$, $C = -0.0202$, $t_c = 1987.5335$, $\phi = 5.0664$, $w = 29.6560$, $\beta = 0.7604$.

1.5 Conclusion and Outlook

As the president of the Federal Reserve Bank of New York, *William C. Dudley* emphasized in 2010 [8]: "... what I am proposing is that we try - try to identify bubbles in real time, try to develop tools to address those bubbles, try to use those tools when appropriate to limit the size of those bubbles and, therefore, try to limit the damage when those bubbles burst". To prevent from the devastating results of bubbles, we should develop new tools, methods or models. Namely, we should understand their dynamics. For this purpose, bubbles could be represented as a mathematical object, so advantages of *optimization*, *pattern recognition tools* and *machine learning* can be used as different solution approaches to detect, forecast and control their size, shape and position.

Here, we suggest a new and pioneering mathematical approach where the logarithmic price process is represented with ellipsoids. In fact, the process is divided into intervals and each interval is described with an ellipsoid. On the basis of this approach, a bubble is dealt with geometrically. By observing the volume of the ellipsoids, a bubble can be detected or the behaviour of the volume change can be used as an *early warning system* against destructive outcomes of the bubbles. After identifying and constructing ellipsoids throughout the price process, they can be clustered; each cluster may depict a bubble structure with fitting a covering ellipsoid.

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